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TECHNICAL NOTES

Radiative transfer in a plane-parallel medium with space-dependent albedo, $\omega(x)$

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ANALYSIS

CONSIDERABLE amount of work is available in the literature for solving radiation transfer in a plane-parallel medium with constant single-scattering albedo; but it appears that only one work [1] has been reported that treats an exponential variation of albedo. In this note we generalize the approach used in ref. [2] to solve the radiation problem involving space-dependent albedo $\omega(x)$, as expressed as a polynomial in the form

$$\omega(x) = \sum_{k=0}^{K} D_k x^k.$$

We consider the following radiation problem for an absorbing, emitting, isotropically scattering, inhomogeneous plane-parallel slab of optical thickness 'a'

$$\mu \frac{\partial I(x,\mu)}{\partial x} + I(x,\mu) = S(x) + \frac{\omega(x)}{2}\int_{-1}^{1} I(x,\mu')\,\mathrm{d}\mu', \quad \text{in} \quad 0 < x < a, \quad -1 \leqslant \mu \leqslant 1 \qquad \text{(1a)}$$

$$I^+(0,\mu) = (1-\rho_0)f_1(\mu) + \varepsilon_1 \frac{n^2\sigma T_1^4}{\pi} + 2\rho_1 \int_0^1 I^-(0,-\mu')\mu'\,\mathrm{d}\mu', \quad \mu > 0 \qquad \text{(1b)}$$

$$I^-(a,-\mu) = (1-\rho_3)f_2(\mu) + \varepsilon_2 \frac{n^2\sigma T_2^4}{\pi} + 2\rho_2 \int_0^1 I^+(a,\mu')\mu'\,\mathrm{d}\mu', \quad \mu > 0. \qquad \text{(1c)}$$

Here x is the optical variable, μ is the cosine of the angle between the positive x-axis and the direction of the radiation intensity, n is the refractive index of the medium, and $S(x)$ is a source term and is related to the temperature $T(x)$ of the medium by $S(x) = [1-\omega(x)]n^2\sigma T^4(x)/\pi$, with σ being the Srefan–Boltzmann constant. In addition, the functions $f_i(\mu)$ and $\varepsilon_i n^2 \sigma T_i^4/\pi$, $i = 1$ or 2, represent respectively the intensities of the externally incident radiation and the emission of radiation from the walls due to their temperatures. Here ε_i ($i = 1, 2$) and ρ_i ($i = 0, 1, 2, 3$) are the diffuse emissivities and reflectivities of the boundary surfaces, respectively.

The incident radiation, $G(x)$, is now defined as

$$G(x) = 2\pi \int_{-1}^{1} I(x,\mu)\,\mathrm{d}\mu = 2\pi\left\{\int_0^1 I^+(x,\mu)\,\mathrm{d}\mu + \int_0^1 I^-(x,-\mu)\,\mathrm{d}\mu\right\}. \qquad (2)$$

Then the radiation problem (1) is formally solved for $I^+(x,\mu)$ and $I^-(x,-\mu)$ as described in ref. [3] and the results are introduced into equation (2). The following integral equation for $G(x)$ is obtained

$$G(x) = Y(x) + \int_0^a \omega(x')K(x,x')G(x')\,\mathrm{d}x' \qquad (3)$$

where the functions $K(x,x')$ and $Y(x)$ are the same as those given in ref. [2] for the case of constant albedo.

To solve equation (3), we represent $G(x)$ in a polynomial of degree N in the x variable

$$G(x) = \sum_{n=0}^{N} C_n x^n \quad \text{in} \quad 0 < x < a \qquad (4)$$

where C_n's are the unknown expansion coefficients. The Galerkin method is now applied by following the procedure outlined in ref. [2] to yield $N+1$ algebraic equations for the determination of $N+1$ unknown coefficients, expressed in the matrix form as

$$[b_{mn}]\{C_n\} = \{d_m\}, \quad m, n = 0, 1, 2, \ldots, N \qquad (5)$$

where $\{C_n\}$ is the column vector of the unknown expansion coefficients C_n. The elements of the square matrix $[b_{mn}]$ and the column vector $\{d_m\}$ are given respectively by

$$b_{mn} = \frac{a^{m+n+1}}{m+n+1} - \sum_{k=0}^{K} D_k \int_0^a \int_0^a K(x,x')x^m x'^{n+k}\,\mathrm{d}x\,\mathrm{d}x' \qquad (6)$$

$$d_m = \int_0^a Y(x)x^m\,\mathrm{d}x. \qquad (7)$$

Equations (5) provide $N+1$ algebraic equations for the determination of $N+1$ unknown expansion coefficients C_n. It is to be noted that the matrix $[b_{mn}]$ is independent of the boundary surface intensities $A_i = \varepsilon_i n^2 \sigma T_i^4/\pi$, $f_i(\mu)$, ($i = 1, 2$) and the source term $S(x)$.

All the integrations appearing in the elements b_{mn} are performed analytically and the resulting expression for b_{mn} is determined as

$$b_{mn} = \frac{a^{m+n+1}}{m+n+1} - \sum_{k=0}^{K} D_k\{\tfrac{1}{2}T_{m,n+k} + \beta[\rho_1 T_m T_{n+k} + \alpha_1\rho_2(T_m T^*_{n+k} + T^*_m T_{n+k}) + \rho_2 T^*_m T^*_{n+k}]\} \quad (8)$$

where

$$\beta = \frac{1}{1-\alpha_1\alpha_2}, \quad \alpha_1 = 2\rho_1 E_3(a), \text{ and } \quad \alpha_2 = 2\rho_2 E_3(a) \quad (9)$$

and $T_{i,j}$, T_i, T^*_i and d_m are defined in ref. [2].

Once C_n's are determined from the solution of equations (5), $I^+(x,\mu)$ and $I^-(x,-\mu)$ anywhere in the medium are given by

$$I^+(x,\mu) = [(1-\rho_0)f_1(\mu) + A_1 + 2\rho_1 K_1]\,e^{-x/\mu} + \frac{1}{\mu}\int_0^x S(x')\,e^{-(x-x')/\mu}\,dx' + \frac{1}{4\pi}\sum_{k=0}^{K} D_k \times \sum_{n=0}^{N} C_n(n+k)!\left\{(-1)^{n+k+1}\,e^{-x/\mu}\,\mu^{n+k} + \sum_{j=0}^{n+k}\frac{(-1)^j x^{n+k-j}}{(n+k-j)!}\mu^j\right\}, \quad \text{for } \mu > 0 \quad (10a)$$

and

$$I^-(x,-\mu) = [(1-\rho_3)f_2(\mu) + A_2 + 2\rho_2 K_2]\,e^{-(a-x)/\mu} + \frac{1}{\mu}\int_x^a S(x')\,e^{-(x'-x)/\mu}\,dx' + \frac{1}{4\pi}\sum_{k=0}^{K} D_k \times \sum_{n=0}^{N} C_n(n+k)!\sum_{j=0}^{n+k}\frac{\mu^j}{(n+k-j)!} \times (x^{n+k-j} - a^{n+k-j}\,e^{-(a-x)/\mu}), \quad \text{for } \mu > 0. \quad (10b)$$

The forward and backward radiation heat fluxes, $q^+(x)$ and $q^-(x)$, anywhere in the medium, are determined, respectively, from

$$\frac{q^+(x)}{2\pi} = (1-\rho_0)\int_0^1 f_1(\mu)\,e^{-x/\mu}\,\mu\,d\mu + (A_1 + 2\rho_1 K_1)E_3(x) + \int_0^x S(x')E_2(x-x')\,dx' + \frac{1}{4\pi}\sum_{k=0}^{K} D_k\sum_{n=0}^{N} C_n(n+k)!\left[(-1)^{n+k+1}E_{n+k+3}(x) - \sum_{j=0}^{n+k}\frac{x^{n+k-j}}{(n+k-j)!}\frac{(-1)^{j+1}}{j+2}\right] \quad (11)$$

and

$$\frac{q^-(x)}{2\pi} = (1-\rho_3)\int_0^1 f_2(\mu)\,e^{-(a-x)/\mu}\,\mu\,d\mu + (A_2 + 2\rho_2 K_2)E_3(a-x) + \int_x^a S(x')E_2(x'-x)\,dx' + \frac{1}{4\pi}\sum_{k=0}^{K} D_k\sum_{n=0}^{N} C_n\sum_{j=0}^{n+k}\frac{(n+k)!}{(n+k-j)!}\times\left[\frac{x^{n+k-j}}{j+2} - a^{n+k-j}E_{j+3}(a-x)\right] \quad (12)$$

where K_1 and K_2 are given by

$$K_1 = \beta\left\{\alpha_2(1-\rho_0)\int_0^1 f_1(\mu)\,e^{-a/\mu}\,\mu\,d\mu + (1-\rho_3)\int_0^1 f_2(\mu)\,e^{-a/\mu}\,\mu\,d\mu + \alpha_2 E_3(a)A_1 + E_3(a)A_2 + \int_0^a [E_2(x) + \alpha_2 E_2(a-x)]S(x)\,dx + \frac{1}{4\pi}\sum_{k=0}^{K} D_k\sum_{n=0}^{N} C_n(T_{n+k} + \alpha_2 T^*_{n+k})\right\} \quad (13)$$

Table 1. Effects of spacial variation of albedo, $\omega(x)$, on hemispherical reflectivity and transmissivity of a slab with $a = 1$, and transparent boundaries

$\omega(x)$	ω_{ave}	Isotropic incidence R	Isotropic incidence Γ	Normal incidence R	Normal incidence Γ
Linear variation of albedo					
x	0.5	0.064838	0.313115	0.058927	0.464561
$0.1+0.8x$	0.5	0.076897	0.310771	0.065708	0.459754
$0.2+0.6x$	0.5	0.089796	0.308978	0.073084	0.455534
$0.3+0.4x$	0.5	0.103594	0.307712	0.081084	0.451865
$0.4+0.2x$	0.5	0.118358	0.306959	0.089747	0.448714
0.5	0.5	0.134165	0.306709	0.099119	0.446058
$0.6-0.2x$	0.5	0.151104	0.306959	0.109253	0.443881
$0.7-0.4x$	0.5	0.169279	0.307712	0.120212	0.442169
$0.8-0.6x$	0.5	0.188806	0.308978	0.132068	0.440916
$0.9-0.8x$	0.5	0.209824	0.310771	0.144908	0.440122
$1-x$	0.5	0.232491	0.313115	0.158831	0.439791
Quadratic variation of albedo					
$0.2+0.2x+0.6x^2$	0.5	0.081072	0.311079	0.067234	0.459948
$0.4-0.2x+0.6x^2$	0.5	0.108266	0.307972	0.082862	0.451945
$0.8-x+0.6x^2$	0.5	0.175240	0.307972	0.122663	0.442115
$1-1.4x+0.6x^2$	0.5	0.216635	0.311079	0.147811	0.440041

Table 2. Angular distribution of radiation in a slab with $\omega(x) = x/a$ and transparent boundaries

a	θ	$x=0$ I^-	$x=a/2$ I^-	$x=a/2$ I^+	$x=a$ I^+	NT	a	θ	$x=0$ I^-	$x=a/2$ I^-	$x=a/2$ I^+	$x=a$ I^+	NT
(a) Isotropic incidence							(b) Normal incidence						
	0	0.0204	0.0154	0.9570	0.9258			0	0.0038	0.0030	0.9523	0.9088	
	15	0.0210	0.0160	0.9555	0.9234			15	0.0039	0.0031	0.0011	0.0041	
	30	0.0233	0.0177	0.9506	0.9151			30	0.0044	0.0034	0.0012	0.0045	
0.1	45	0.0280	0.0216	0.9399	0.8974	6	0.1	45	0.0053	0.0041	0.0014	0.0055	6
	60	0.0382	0.0300	0.9162	0.8593			60	0.0072	0.0058	0.0020	0.0076	
	75	0.0655	0.0552	0.8456	0.7533			75	0.0123	0.0106	0.0038	0.0139	
	90	0.0000	0.2257	0.2257	0.3966			90	0.0000	0.0414	0.0414	0.0789	
	0	0.0589	0.0529	0.6370	0.4398			0	0.0176	0.0177	0.6145	0.3909	
	15	0.0599	0.0544	0.6273	0.4287			15	0.0179	0.0182	0.0082	0.0236	
	30	0.0627	0.0590	0.5957	0.3940			30	0.0186	0.0197	0.0090	0.0254	
1.0	45	0.0671	0.0681	0.5333	0.3319	8	1.0	45	0.0197	0.0227	0.0106	0.0288	8
	60	0.0712	0.0846	0.4196	0.2405			60	0.0204	0.0281	0.0138	0.0346	
	75	0.0635	0.1132	0.2211	0.1491			75	0.0169	0.0370	0.0207	0.0437	
	90	0.0000	0.1187	0.1187	0.1273			90	0.0000	0.0355	0.0355	0.0475	
	0	0.0136	0.0071	0.0943	0.0126			0	0.0051	0.0050	0.0882	0.0110	
	15	0.0136	0.0072	0.0874	0.0114			15	0.0050	0.0050	0.0061	0.0043	
	30	0.0137	0.0074	0.0681	0.0086			30	0.0049	0.0052	0.0063	0.0042	
5.0	45	0.0136	0.0079	0.0415	0.0060	12	5.0	45	0.0046	0.0054	0.0065	0.0040	10
	60	0.0130	0.0084	0.0188	0.0046			60	0.0041	0.0057	0.0067	0.0037	
	75	0.0106	0.0091	0.0111	0.0039			75	0.0028	0.0060	0.0066	0.0032	
	90	0.0000	0.0098	0.0098	0.0013			90	0.0000	0.0063	0.0063	0.0024	

$$K_2 = \beta\left\{(1-\rho_0)\int_0^1 f_1(\mu)\,e^{-a/\mu}\,\mu\,d\mu + \alpha_1(1-\rho_3)\int_0^1 f_2(\mu)\,e^{-a/\mu}\,\mu\,d\mu + E_3(a)A_1 + \alpha_1 E_3(a)A_2 + \int_0^a [\alpha_1 E_2(x) + E_2(a-x)]S(x)\,dx + \frac{1}{4\pi}\sum_{k=0}^{K} D_k \sum_{n=0}^{N} C_n(\alpha_1 T_{n+k} + T^*_{n+k})\right\}. \quad (14)$$

In the above relations $E_n(x)$'s represent exponential integral functions.

RESULTS AND DISCUSSION

In order to illustrate the application of the foregoing analysis, we consider a radiation problem for a slab with transparent boundaries (i.e. $\rho_i = 0$ for $i = 0, 1, 2, 3$). We assume negligible emission of radiation by the medium and the walls [i.e. $S(x) = A_1 = A_2 = 0$], and no externally incident radiation at the boundary surface $x = a$ (i.e. $f_2(\mu) = 0$). Then the only source in the medium is due to the externally incident radiation $f_1(\mu)$ at the boundary surface $x = 0$. We consider both the isotropic and the normal incidence of unit intensity.

In many engineering applications, when the albedo varies within the medium, generally an average value is used for radiation calculations. Here we examine the error involved in

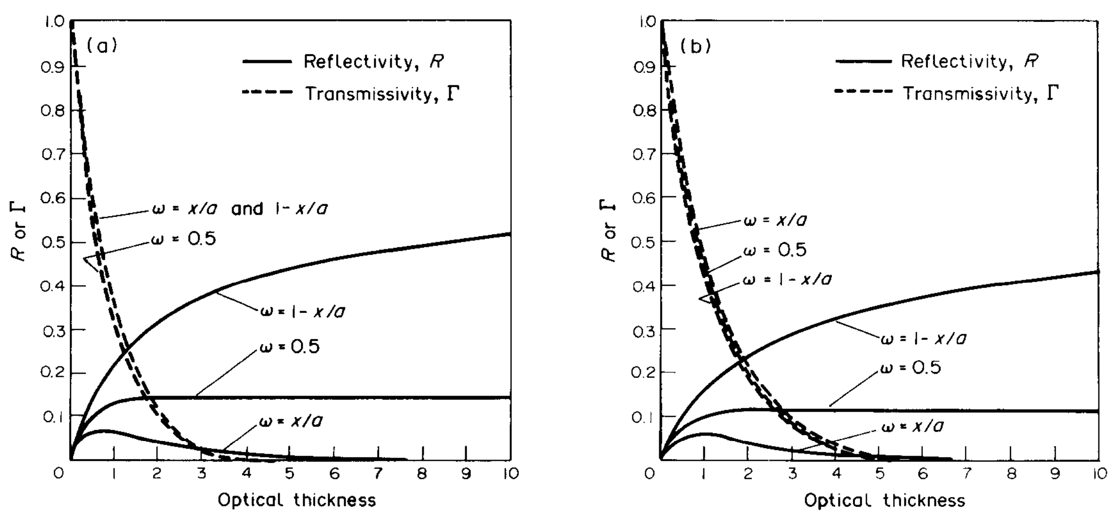

FIG. 1. Effects of spacial variation of albedo on reflectivity and transmissivity of a slab: (a) isotropic incidence; (b) normal incidence.

transmissivity and reflectivity, when the average value of albedo is used instead of the actual variation of albedo within the medium. To investigate this matter, we consider several different spacial variations of albedo within the medium such that for each case the average value of $\omega(x)$ over the medium is equal to $\omega_{ave} = 0.5$. Table 1 lists the results obtained from such calculations for both linear and quadratic variations of albedo for an optical thickness of $a = 1$. It appears that the reflectivity of the slab varies significantly with the variation in the distribution of $\omega(x)$, whereas the change in transmissivity is very small. For example, for the case of $\omega(x) = x$ and isotropic incidence, the use of arithmetic average for ω overestimates reflectivity about 107%, but the transmissivity is underestimated only by about 2%. In the case of normal incidence, these errors are 68 and 4%, respectively.

This same trend continues for all optical thicknesses as shown in Fig. 1. The transmissivity curves are very close to each other indicating a weak dependence of transmissivity on the spacial variation of albedo. As expected, the transmissivity approaches zero as the optical thickness becomes larger. On the other hand the reflectivity curves for $\omega(x) = x/a$ and $\omega(x) = 1-x/a$ are far apart from each other, especially at larger optical thicknesses, indicating strong dependence of reflectivity on the spacial distribution of the albedo. A similar trend is apparent for the case of normal incidence as shown in Fig. 1(b).

The present approach is also applicable for the exponential variation of albedo in the form $\omega(x) = \omega_0 \, e^{-x/s}$, if the exponential function is represented by its polynomial expansion. This is demonstrated by comparing our results with those obtained in ref. [1] using the F_N method, for the case of $s = 100$. We used the first ten terms of the polynomial expansion of the exponential function, which involves an error less than 10^{-16} in albedo for $a \leq 10$. The agreement between the two results was excellent.

In Table 2, we present the angular distribution of radiation intensity for $\omega(x) = x/a$ at three different locations in the medium (i.e. at $x = 0$, $a/2$ and a) for different optical thicknesses for the cases of both isotropic and normal incidence. Here we list in the last column the number of terms, NT, used in the expansion to achieve convergence for a specified degree of accuracy. The functional form of the albedo, $\omega(x)$, did not seem to have much effect on the number of terms required to achieve a specified accuracy.

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Effect of inert regions on local mass transfer rate measurements using the limiting diffusion current technique—case of Poiseuille type flow

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NOMENCLATURE

- c concentration
- D diffusion coefficient
- H height of the channel
- h grid size, x-direction
- k grid size, y-direction
- L length of the nonconducting segment
- Pe Peclet number, $(Re\,Sc) = Hu_{av}/D$
- R length of conducting region/length of nonconducting region
- Sc Schmidt number, $\mu/\rho D$
- u velocity
- u_{av} average velocity
- x, y x-, y-coordinate.

Superscript
- $*$ dimensionless quantity represented by equation (5).

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1. INTRODUCTION

Most widely used in recent years for fluid–solid mass transfer studies is the limiting current technique, which measures a current density at a cathode where reduction of ferricyanide takes place. The details of the limiting current technique may be found elsewhere [1–3].

In order to understand local characteristics of mass transfer around the solid surface it becomes necessary to measure local